Trading Favors: Optimal Exchange and Forgiveness

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August 2010 (First Draft: June 2005)

Abstract

How is cooperation without immediate reciprocity sustained in a long term relationship? We study the case of two players who have privately observable opportunities to provide favors, and where the arrival of these opportunities is a Poisson process. Favors provided by a player give her an entitlement to future favors from her partner. We characterize and solve for the Pareto frontier of Public Perfect Equilibria (PPE) and show that it is self-generating. This guarantees that the equilibrium is renegotiation proof. We also find that optimal PPE have two key characteristics: 1) the relative price of favors decreases with a player's entitlement and 2) the disadvantaged player's utility increases over time during periods of no trade, so in the optimal equilibria there is forgiveness. We show that these two features allow for considerably higher payoffs than a standard *chips* mechanism. The game is in continuous time with Poisson jumps. We provide a characterization of the Pareto frontier via an Integro-differential equation. *Keywords. Repeated games; imperfect monitoring; dynamic insurance; partnerships; forgiveness; medium of exchange.*

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1 Introduction

How is cooperation without immediate reciprocity sustained in a long term relationship? Consider the following example: Two firms are engaged in a joint venture. At random times one of them finds a discovery; if disclosed, the discovering firm's payoffs will be lower but total payoffs higher. In the absence of immediate reciprocity, will the expectation of future cooperation induce disclosure? And how much cooperation can be supported? In the case of perfect monitoring -where the arrival of the discovery is jointly observed- full cooperation can be supported with trigger strategies when discounting is not too strong. But in the case of imperfect monitoring -where the arrival of the discovery is privately observed- the first best cannot be supported and the above questions have no definite answer. This paper attempts to fill this gap.

The canonical model of money as a medium of exchange *implements* an efficient social arrangement when there is an absence of opportunities for immediate reciprocity.[14] The role of private information in such settings has been emphasized as a foundation for the use of money as a record-keeping device.[16, 25][1] In small group/long term relationships, such as in many partnerships, fiat money is rarely used on a routine basis to induce cooperation. In the absence of immediate reciprocity, the anticipation of future compensation in the form of favors by other participants is central for cooperation. *Inside money,* has been used as a recordkeeping device, as in the well known baby-sitter exchange program of Capitol Hill co-op discussed in [24]. [2] Our analysis derives the constrained efficient arrangement and the implied properties of an optimal record-keeping device.

The setup is as follows.[3] Two players interact indefinitely in continuous time. At random arrival times, one player has the possibility of providing a benefit b to the other player at a cost $c < b$.[4] Following [19] we call this a *favor.* This

[1] This idea goes back to [4]: it is "the costliness of information about the attributes of goods available for exchange that induces the use of money in an exchange economy."

[2] The Capital Hill coop issued scrip–pieces of paper equivalent to half hour of baby-sitting time. "Baby sitters would receive the appropriate number of coupons directly from the baby sittees. This made the system self-enforcing: Over time, each couple would automatically do as much baby-sitting as it received in return."[17]

[3] Our setup is identical to [19], with very minor modifications.

[4] The motivating example can be easily accommodated by letting b denote the value to firm 2 when firm 1 shares information and c the decrease in firm 1's profits when it discloses the information instead of keeping it secret.

opportunity is privately observed, so a player will be willing to do this favor only if this gives her an entitlement to future favors from the other player. Formally, this model is a repeated game with incomplete monitoring with random time intervals (given by the arrival of favors). We characterize and solve for the Pareto frontier of Public Perfect Equilibria (PPE) of that game.

[19] considers a simple class of PPE, which we will call the *simple chips mechanism.* As in the Capitol Hill coop. example, chips are used as a recordkeeping device.[5] Both players start with K chips each. Whenever one receives a favor, she gives the other player a chip. If a player runs out of chips (so the other one has $2K$), she receives no favor until she grants one to the other player and obtains a chip in exchange. Given that the arrival of a favor is private information, giving favors must be voluntary, and motivated by the claim to future favors. Incentive compatibility and discounting then put a limit on the number of chips that can exist in the economy. This is obviously a convenient and straightforward mechanism. However, it has two special features which suggest that there is room for improvement. First, the exchange rate is always one (current) for one (future) favor, so it is independent of the distribution of chips. Due to discounting, a player that is entitled to many future favors will value a marginal favor less. This suggests that the rate of exchange (or relative price) of favors should depend on current entitlements. Secondly, entitlements do not change unless a favor is granted (e.g. chips do not *jump* from one player to the other). This is a special feature that rules out the possibility of appreciation or depreciation of claims. As we show in the paper, relaxing these two features allows for higher payoffs.

Our analysis proceeds in several steps. As usual in the literature, the recursive approach introduced by [3] is used. We adapt it to our continuous time framework and first establish that the set of Pareto optimal PPE is self-generating. This is not generally true in games of private information and relies on some special features of our formulation which we discuss. Moreover, it also guarantees that the equilibrium is renegotiation proof. As a result, the recursive formulation reduces to a one dimensional dynamic programming problem which is solved by a simple algorithm. Optimal PPE have two key features: 1) The relative price of favors decreases with a player's entitlement. So starting from an initial symmetric point (the analogue of players having an equal number of

[5] This type of mechanism is used by [23] in a repeated auction with incomplete information.

chips) if a player receives any number of consecutive favors, he must give back a considerably greater number of favors before returning to the initial point. 2) The entitlements change over time even in periods with no trade. We solve the model numerically for a large set of parameter values and find that the gains relative to the chips mechanism can be quite large (in some cases over 30% higher). Interestingly, in all our numerical simulations the disadvantaged player's utility increases over time during periods of no trade, so in the optimal equilibrium there is forgiveness.

Our model is a continuous time, repeated game with imperfect monitoring. This is a class of games that has not been previously analyzed. An exception is [21], which in independent recent work studies games within this class where the stochastic component follows independent diffusion processes and provides a differential equation that characterizes the boundary of the set of PPE. Our model does not fit exactly in that class since our stochastic process is a jump process (Poisson arrivals), yet we also derive a differential formulation to characterize the boundary of the set of PPE.[6] In addition we establish that the set of Pareto optimal PPE is self-generating, which as far as we know is a new result in the literature on repeated games with imperfect monitoring.

Our model is also connected to a large literature on dynamic incentives for risk sharing. Indeed, if we reinterpret the favor as a consumption good, the benefit as the utility for the receiver and the cost as the foregone utility for the player giving the favor, the utilitarian solution calls for allocating the good to the agent with highest marginal utility for consumption. Most of this literature in macroeconomics, starting with the seminal paper [10] by Green, consider an environment with a large (continuum) number of players with privately observed endowments (see also [6].) Kocherlakota [15] considers the question of risk sharing between two agents without commitment, but complete information. Our model is in the intersection of these two literatures, bringing *private information* to a case of *bilateral risk sharing.*

In our model there is a lack of double coincidence of needs, as players cannot immediately return the favors received. As suggested in a related paper [1], players

[6] In a recent paper, [22] study PPE in games with frequent actions with an information structure that nests Brownian motion and Poisson jumps. Results in their paper suggest that the PPE set for the continuous time Poisson game that we consider approximates the PPE of their discrete holding-time game when actions can be changed frequently. We provide a similar approximation result.

give favors trusting that the receiver will have incentives to reciprocate in the future. The lack of observability of opportunities for exchange is a complicating factor that limits the possibilities of exchange. However, it can be easily shown that as the discount rate goes to zero (or the frequency of trading opportunities goes to infinity) the cost of this informational friction disappears.

The paper is organized as follows. Section 2 describes the model. Section 3 describes in more detail the simple chips mechanism. Section 4 develops the recursive formulation. Section 5 derives the differential approach and provides an algorithm, proves a folk theorem and draws connections to the literature on dynamic games. Section 6 derives the results on forgivness. Section 7 provides an alternative state representation of the game in terms of accounting of favors owed by players. Section 8 gives numerical results and compares the performance of our mechanism with the chips mechanism. Section 9 maps our mechanism into a generalized chips and draws some links to the literature on optimal inflation. Section 10 concludes.

2 Model

We analyze an infinite horizon, two-agent partnership. Time is continuous and agents discount future utility at a rate r. There are two symmetric and independent Poisson processes -one for each agent- with arrival rate α representing the opportunity of producing a favor. We assume that favors are perfectly divisible, so partners can provide fractional favors.[7] Agents' utilities and costs are linear in the amount of favors exchanged. The cost per unit of a favor is c and the corresponding benefit to the other player $b > c$. Letting x_i represent a favor granted by player i and x_j a favor received, the utility for player i is given by:

$$U_i(x_i, x_j) = -cx_i + bx_j.$$

Since arrivals are Poisson and independent, only one player is able to grant a favor at a point in time, so $x_i(t) > 0$ implies $x_j(t) = 0$. Arrivals are privately observed by each player, so the ability to provide a favor is private information. Since the cost of providing a favor is less than the generated benefit, it is socially

[7] Alternatively, given our assumption of linear utilities, we can assume there is public randomization for the provision of favors.

optimal for agents to always grant favors. Indeed, in the absence of informational constraints, a public perfect equilibrium would exist that achieves this optimum through a simple Nash reversion strategy: an agent grants favors whenever she can, as long as her partner has done so in the past, and stops granting favors whenever her partner has defected. This equilibrium can be supported for

$$c < \left(\frac{\alpha}{r+\alpha}\right) b$$

The problem with private information is that an agent only observes whether her partner has provided a favor or not, but is unable to detect a deviation where her partner has passed the opportunity to do a favor. The question then becomes: how to ensure the maximum cooperation and exchange of favors between agents given these informational constraints?

3 A Simple *chips* Mechanism

In [19], Mobius considers equilibria of a simple class. The equilibrium proposed is Markov perfect, where the state variable is the difference k between the number of favors granted by agent 1, and those granted by agent 2. We call this a simple chips mechanism (SCM). For $-K \leq k \leq K$, the agents obey the following strategies:

- Agent 1 grants favors if $k < K$, and stops granting favors if $k = K$
- Agent 2 grants favors if $k > -K$, and stops granting favors is $k = -K$

It is obvious that the magnitude of K is crucial in determining the expected payoffs of players. Since it is efficient to have favors done whenever possible, an efficiency loss occurs when agents reach the boundaries K and $-K$, where only the indebted agent is granting favors. The larger K is, the lower is the incidence of this situation, and the larger are the expected payoffs of the agents. On the other hand, since one favor done today is rewarded by the promise of exactly one favor in the future, K cannot be infinitely large. To understand how K is determined, note that because of discounting the marginal value of the right to an extra favor diminishes with the current entitlement of favors the player has; $2K$ is the largest number such that this marginal value exceeds the cost c.

The SCM is very easy to implement. Moreover it is asymptotically efficient (as $\alpha/r \to \infty$). To see this, note that in the long run the distribution over the states $k = \{-K, -K+1, ..., 0, 1, ..., K-1, K\}$ is uniform[8], so the probability that a favor is not granted is $1/K$. It is easy to verify that $K \to \infty$ as $\alpha/r \to \infty$, so this probability goes to zero.

It is also easy to derive bounds on V_K, incentive compatibility for the *last* favor requires $V_K - V_{K-1} \geq c$ and given

$$rV_K = \alpha b - \alpha\left(V_K - V_{K-1}\right),$$

it follows that $\frac{r}{\alpha}V_K \leq (b-c)$. Note that this is bounded away from $b - \frac{c^2}{b}$, the highest value for a player in the set of feasible and individually rational payoffs. Though this might seem at odds with our previous statement, it is not. The bound $(b-c)$ is achieved asymptotically as $r/\alpha \to 0$ *regardless of the distribution of chips!*

Proposition 1. *In the optimal chips mechanism, $\frac{r_n}{\alpha_n}V_k^n \to b-c$ for all $-K_n \leq k \leq K_n$ as $\frac{r_n}{\alpha_n} \to 0$.*

The chips mechanism cannot support inequality in this limit, but achieves the first best symmetric payoffs. This is in contrast to Perfect Public equilibria where, as we show in Section 5.2, the folk theorem holds.

We now derive necessary and sufficient conditions for trade to be possible under a chips mechanism. It is easy to see that if a *one total chip* mechanism -where either a player has a chip or not- cannot be implemented, then neither can any other number of chips. In this one chip mechanism a player can be in either of two states $\{0,1\}$, reflecting whether she has a chip or not, respectively. The value functions for these cases are:

$$rv_0 = -\alpha c + \alpha\left(v_1 - v_0\right)$$

$$rv_1 = \alpha b - \alpha\left(v_1 - v_0\right)$$

Subtracting the first equation from the second one we get

$$v_1 - v_0 = \frac{\alpha}{r+2\alpha}\left(b+c\right)$$

[8] The flow in and out of interior states is equal for all states. The flow into corner states is half of that (can only be approached from one side) but the flow out is also half.

and substituting in the incentive constraint $c \leq (v_1 - v_0)$ gives:

$$c \leq \frac{\alpha}{r + \alpha} b.$$

Remarkably, this is the same condition that is needed to support cooperation in the case where there is no private information!

There are two special features of the chips mechanism, which suggest that there is room for improvement. The first special feature is that the rate of exchange of current for future favors is the same (equal to one) regardless of entitlements. Relaxing this constraint could reduce the region of inefficiency for two reasons. First, consider the case where the state is K so agent one has the maximum entitlement of favors. As we argued before, at this point agent one's marginal value to an entitlement of an extra favor is lower than c. But there is still room for incentives if agent two were to promise more than one favor in exchange. Moreover, in the SCM the incentive constraints only bind at the extremes, but are slack in between, where the marginal value to future favors exceeds c. A lower rate of exchange could allow to expand the number of possible favors.

The second special feature is that agents' continuation values do not change unless an agent does a favor. This is restrictive, and rules out the possibilities of appreciation (charging interest) or depreciation (forgiveness) of entitlements and punishment in case "not enough" cooperation is observed.

In the following section, strategies are not limited to a particular scheme. Instead, we characterize the optimal Perfect Public equilibria.

4 Characterizing the Optimal Perfect Public Equilibrium

4.1 Definition

The game described above falls in the class of repeated games with imperfect monitoring. As usual in this literature, we restrict our analysis to Public Perfect Equilibria (PPE), where strategies are functions of the public history only and equilibrium is perfect Bayesian.

A public history up to time t, denoted by h^t, consists of agents' past favors including size and date. A strategy $x_{it} : h^t \rightarrow [0,1]$ for player i specifies for every history and time period the size of favor the agent grants if the opportunity to do so arises. A public perfect equilibrium is a pair of strategies $\{x_{1t}, x_{2t}\}_{t \geq 0}$ that constitute a perfect Bayesian equilibrium. To analyze this game, we consider a recursive representation following the formulation in [3] (APS). Let

$$\mathbf{V}^* = \{(v,w) \,|\, \exists \text{ PPE that achieves these values}\},$$

be the set of PPE values, where v and w denote player 1 and 2's values respectively. This set is nonempty, as it contains the trivial equilibrium with no favors and values $(0,0)$. The set V^* is a subset of the set of feasible and individually rational payoffs, which is given in figure(4.1).

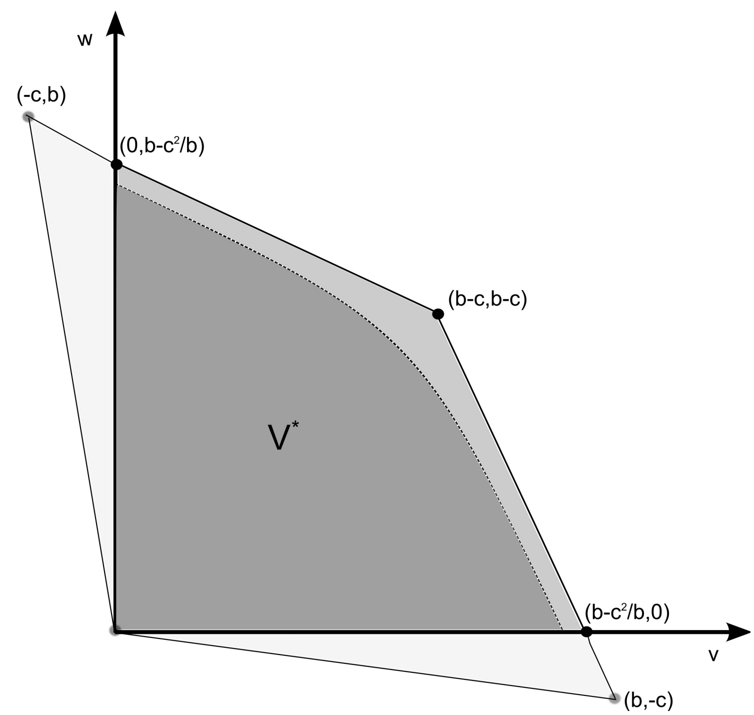

Figure 4.1: Feasible and Equilibrium Payoffs

The set of feasible payoffs is given by the whole outer polygon. Values have been multiplied by r/α so they are expressed as flow-equivalents. The extreme points are given by the vectors $(-c,b)$, $(b,-c)$, $(b-c,b-c)$, $(0,0)$. The first two correspond to the situation where only one player is giving favors and doing so whenever possible; the third vector corresponds to full cooperation by both players and the last one to no cooperation. The set is obtained by convex combinations of these points and is derived by combining the endpoints over time. Individually rational payoffs are found by restricting this set to the positive orthant. The set V^* is a proper subset of this set.

4.2 Factorization and the Equilibrium set[9]

To characterize the set V^* we follow the general idea in APS [3], which decompose (factorize) equilibrium values into strategies for the current periods and continuation values for each possible end of period public signal. The difficulty in our case is that there is no *current period* in our model.

Adapting the approach in APS, equilibrium payoffs can be factorized in the following way. Let t denote the (random) time at which the next favor occurs. Consider any $T > 0$. Factorization is given by functions $x_1(t), x_2(t), v_1(t), v_2(t), w_1(t), w_2(t)$ and values v_T, w_T with the following interpretation. If the first favor occurs at time t and is given by player i, then $x_i(t) \in [0,1]$ specifies the size of the favor, $v_i(t)$ the continuation value for player one and $w_i(t)$ the continuation value for player two, where for each t the vector $(v_i(t), w_i(t)) \in \mathbf{V}^*$. If no favor occurs until time T, the respective continuation values are v_T, w_T. The strategies and continuation values give the following utility to player one:

$$v = \int_0^T e^{-rt} \left\{ \frac{x_2(t)\, b + v_2(t) - x_1(t)\, c + v_1(t)}{2} \right\} p(t)\, dt + e^{-(r+2\alpha)T} v_T$$

where $p(t)$ denotes the density of the first arrival occurring at time t. This is the density of an exponential distribution with coefficient 2α (the total arrival rate). Letting $\beta = e^{-(r+2\alpha)}$ and $z_1(t) = x_2(t)\, b - x_1(t)\, c + v_1(t) + v_2(t)$, the above equation simplifies to:

$$v = \alpha \int_0^T \beta^t z_1(t)\, dt + \beta^T v_T. \tag{4.1}$$

Similarly, letting $z_2(t) = x_1(t)\, b - x_2(t)\, c + w_1(t) + w_2(t)$ one obtains the value

[9]The reader not interested in technical details can skip to Section 5

w for player two:

$$w = \alpha \int_0^T \beta^t z_2(t)\, dt + \beta^T w_T. \tag{4.2}$$

The incentive compatibility condition requires that for all $0 \le t \le T$,

$$v_1(t) - x_1(t)\, c \ge v(t) \equiv \alpha \int_t^T \beta^{s-t} z_1(s)\, ds + \beta^{T-t} v_T \tag{4.3}$$

$$w_2(t) - x_2(t)\, c \ge w(t) \equiv \alpha \int_t^T \beta^{s-t} z_2(s)\, ds + \beta^{T-t} w_T \tag{4.4}$$

The left hand side represents the net utility of giving a favor at time t and the right hand side the continuation utility if the agent passes this opportunity.

Note that $T \le \infty$ is arbitrary in this factorization.[10] For $T = \infty$, the first favor defines a corresponding stopping time τ at which payoffs are realized and a continuation PPE path granting values $(v_i(t), w_i(t)) \in V^*$ is started. For $T < \infty$, the stopping time $\tau_T = \min(\tau, T)$, with the same payoffs and continuation values if the first favor occurs before T and in the contrary, no payoffs accrue and a new path is started at T with values $(v_T, w_T) \in V^*$.

Proposition 2. *The set of equilibrium payoffs V^* is compact and convex.*

Proof. The proof mimics the arguments in Abreu, Pearce and Stacchetti. Take a compact set of continuation payoffs V . Define the set $B_T V$ as all values (v, w) that satisfy conditions (4.1),(4.2),(4.3) and (4.4), where all pairs $(v_i(t), w_i(t)) \in V$, $i = \{1, 2\}$ and $(v_T, w_T) \epsilon V$. We will show that $B_T V$ is compact. It is obviously nonempty by choosing constant values and setting $x_i(t) = 0$ for all $t \in [0, T]$ and $i = 1, 2$. Endow the space of functions mapping $[0, T]$ into V and $[0, T]$ into $[0, 1]$ with the product topology (pointwise convergence.) Because each of the coordinate spaces is compact, so is this product space. Take a sequence $\{v_n, w_n\}_{n=0}^{\infty}$ in $B_T V$ with associated strategies and continuation values $\left\{v_i^n(t), w_i^n(t), x_j^n(t)\right\}_{n=0, i=\{0,1,2\}, j=1,2}^{\infty}$. Let $\{v_i(t), w_i(t), x_j(t)\}_{i=\{0,1,2\}, j=1,2}$ be the limit of a converging subsequence and (v, w) the corresponding initial period values for the players. We must show that $(v, w) \in B_T V$. By the Lebesgue dominated convergence theorem, the integrals in equations (4.1)-(4.4) converge along this subsequence and so do the other coordinate values. Hence all incentive constraints are satisfied. Convexity follows from the linearity of payoffs

[10] In appendix B we show that the largest self-generating set V of the operator B_T is independent of T.

and incentive constraints. More precisely, for any convex set V, $B_T V$ is convex. As in Abreu, Pearce and Stacchetti, the operator B_T is monotone, so starting from the convex and compact set A of feasible and individually rational payoffs, $V^* = \lim_{n\to\infty} B_T^n A = \cap_{n=1}^{\infty} B_T^n A$, since this sequence of sets is decreasing. Given that for each $B_T^n A$ is convex and compact for each n, so is the infinite intersection. $\square$

Let $W(v) = \max\{w | (v, w) \in V^*\}$ denote the boundary of V^*. By the previous Proposition, this function is well defined, continuous and concave. The following proposition establishes that the domain of this function includes zero and establishes a bound on its slope.

Proposition 3. *The domain of the function W is $[0, \bar{v}]$, where $W(0) = \bar{v}$ ($W(\bar{v}) = 0$) and has slope in $\left[\frac{-b}{c}, \frac{-c}{b}\right]$.*

Proof. Take any point $v > 0, W(v)$ such that the slope of W to the right of v is lower than $-c/b$ and consider factorization

$$z(t) = (x_1(t), x_2(t), v_1(t), v_2(t), w_1(t), w_2(t), v_T, w_T).$$

We will show that it is feasible to decrease v and move in direction $-c/b$ in V^*. First note that if $x_2(t) > 0$ in a set of positive measure, an alternative incentive compatible path can be created by setting $\tilde{x}_2(t) = \delta x_2(t)$ for $\delta < 1$ that will give values $v - b\varepsilon, W(v) + c\varepsilon$ for some $\varepsilon > 0$. Suppose then $x_2(t) = 0$ for all $t \leq T$. If $v_T > v$, by choice of v, we can reduce v_T and increase w_T at a rate at least c/b, moving in V^* to the left of v in direction at least c/b. So assume $v_T \leq v$. It follows that

$$\begin{aligned} v\left(1-\beta^T\right) &\leq \alpha \int_0^T \beta^t \left(v_1(t) - cx_1(t) + v_2(t)\right) dt \\ &= \left(1-\beta^T\right) \int_0^T \frac{2\alpha\beta^t \left(\frac{v_1(t)-c+v_2(t)}{2}\right)}{1-\beta^T} dt. \end{aligned}$$

This integral equals $\frac{2\alpha}{r+2\alpha}$ times a weighted sum of $\frac{v_1(t)-c+v_2(t)}{2}$, implying that either $v_1(t) - c$ or $v_2(t) > v$ in a set of positive measure. If $v_2(t) > v$ in a set of positive measure, we can decrease $v_2(t)$ an increase $w_2(t)$ in that set, moving in the desired direction to the left of v. In the contrary, if $v_2(t) \leq v$ for all t, there must exist a set of positive measure where $v_1(t) - c > v$ and the incentive constraint does not bind. Now repeate the procedure described above by decreasing v_1 and increasing w_1 in that set.

We have established that for $v > 0$ it is always possible to move to the left of a point in the boundary in the direction $-c/b$. This proves that the domain must contain zero and the slope is no greater than $-c/b$. By symmetry on the side where $v > w$, the slope is bounded below by $-b/c$. This completes the proof. □

It is interesting to note that the bounds on the slopes are precisely the ones that correspond to the set of feasible and individually rational payoffs. Also note that in contrast to the SCM scheme, the individual rationality constraint binds at a players minimum value.

4.3 Self-generation of the Pareto frontier

In general, payoffs in the Pareto frontier may require *inefficient equilibria* (i.e. equilibria with dominated payoffs) after some histories. As in standard moral hazard problems, as time elapses and no favors are observed, likelihood ratios for cheating become high for both players, so one might expect that *efficient* punishments might require both players to be punished, resulting in continuation values that are below the Pareto frontier. The following proposition shows that this is not needed in our repeated game.

Proposition 4. *The Pareto set of values* $\{(v, w) \in \mathbf{V}^* \text{ such that } w = W(v)\}$ *is self-generating.*

Proof. See the appendix. □

Self-generation implies that any point in the Pareto frontier can be obtained by relying on continuation values that are also in the frontier. This implies that PPE supporting the Pareto frontier are renegotiation proof. This result is not generally true in games of private information where inefficient punishments may be necessary in order to achieve the highest level of cooperation. Consider a discrete time version of our game where in each period three -privately observed-states could occur: with probability $\alpha < \frac{1}{2}$ player one has the opportunity to do a favor, with equal probability it is player's two opportunity and with probabiity $(1-2\alpha)$ none of them does.[11] Then we can write players' values as

$$v = \alpha(b-c) + \beta(\alpha v_1 + \alpha v_2 + (1-2\alpha)v_0) \tag{4.5}$$
$$w = \alpha(b-c) + \beta(\alpha w_1 + \alpha w_2 + (1-2\alpha)w_0)$$

where β is the discount rate. To implement full favors, incentive compatibility for the two players requires

$$\beta(v_1 - v_0) \geq c$$
$$\beta(w_2 - w_0) \geq c$$

Without loss of generality, assume the incentive constraints bind. Take an interior point (v_0, w_0) and consider moving to the frontier in the direction $dv_0 = dw_0$. Using (4.5) and the above incentive constraints it follows that:

$$\begin{aligned} dv &= \beta(1-\alpha)\,dv_0 + \frac{\beta\alpha}{W'(v_2)}dw_0 \\ &\geq \beta(1-\alpha)\,dv_0 - \beta\alpha\frac{b}{c}dw_0 \\ dw &= \beta(1-\alpha)\,dw_0 + \beta\alpha W'(v_1)\,dv_0 \\ &\geq \beta(1-\alpha)\,dw_0 - \beta\alpha\frac{b}{c}dv_0 \end{aligned}$$

This will result in a positive increase in v and w, provided $(1-\alpha) > \alpha b/c$.This shows that for sufficiently small α, (v_0, w_0) must be in the frontier of V^*. On the contrary, for sufficiently large α the opposite is true and interior punishments give greater value. Figure 4.3 gives a graphical representation of this tradeoff. The large light gray triangle connects incentive compatible points $(v_0, w_0), (v_1, w_1), (v_2, w_2)$ which are all on the Pareto frontier. This gives expected continuation values for the players (v^c, w^c), where $v^c = \alpha(v_1 + v_2) + (1-2\alpha)v_0$ and $w^c = \alpha(w_1 + w_2) + (1-2\alpha)w_0$ that lie in the simplex defined by the three vertices of the triangle. More precisely, since the two endpoints have the same weight α, (v^c, w^c) is on the dashed line joining (v_0, w_0) and the midpoint of the base of the triangle, with weights $(1-2\alpha)$ and 2α, respectively.

[11] This approximation can be justified by the results in [?] that establishes that in a game with frequent moves, but where information arrives continuously, the incentive-decomposition gives values that are arbitrarily close to boundary of the equilibrium set as the frequency of moves increases.

The dark grey triangle with vertices $(\tilde{v}_0, \tilde{w}_0), (\tilde{v}_1, \tilde{w}_1), (\tilde{v}_2, \tilde{w}_2)$ gives an alternative incentive compatible vector where $(\tilde{v}_0, \tilde{w}_0)$ is now an interior point, chosen to make the other two points as close to each other as possible. This results in a gain if favors occur,due to strict concavity, at the cost of a lower continuation if no favors occur. For small values of α, the latter scheme results in values (v^c, w^c) close to $(\tilde{v}_0, \tilde{w}_0)$ while the former gives values close to (v_0, w_0) which is obviously higher. But for high values of α, (v^c, w^c) will be close to the midpoints of the base of each triangle, resulting in higher continuation values with interior punishments.

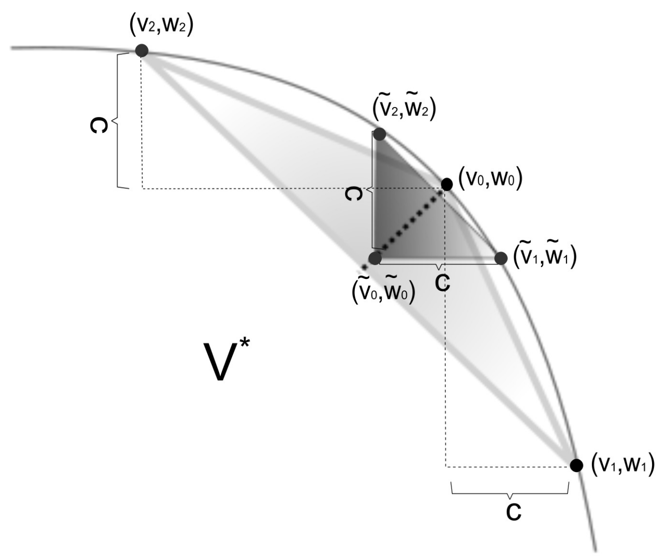

Figure 4.2: Interior vs. boundary punishments

Intuitively, when the time period shrinks (basically to zero in continuous time), the information contained in not observing a signal becomes too imprecise to make it worthwhile to punish: the loss from inefficient punishments becomes geater than the gain from improved incentives.[12]

This self-generation result allows us to write a dynamic programming problem

[12]This is reminiscent of the "good news" case in [2], who point out that the timing of information is critical in designing effective punishments and achieving cooperation in moral hazard problems with continuous time.

defining the frontier $W(v)$ as follows:

$$W(v) = \max_{\{x_i(t), v_i(t), v_T\}} \alpha \int_0^T \beta^t \{x_1(t) b - x_2(t) c + W(v_1(t)) + W(v_2(t))\} dt + \beta^T W(v(T))$$

subject to promise-keeping constraint

$$v = \alpha \int_0^T \beta^t \{x_2(t) b - x_1(t) c + v_1(t) + v_2(t)\} dt + \beta^T v(T))$$

and incentive constraints

$$v_1(t) - x_1(t) c \geq v(t) \equiv \alpha \int_t^T \beta^{s-t} z_1(s) ds + \beta^{T-t} v_T \tag{4.6}$$

$$W(v_2(t)) - x_2(t) c \geq w(t) \equiv \alpha \int_t^T \beta^{s-t} z_2(s) ds + \beta^{T-t} w_T \tag{4.7}$$

where

$$z_1(s) = x_2(s) b - x_1(s) c + v_1(s) + v_2(s)$$
$$z_2(s) = x_1(s) b - x_2(s) c + W(v_1(s)) + W(v_2(s)).$$

The optimal value function W is the largest fixed point of this problem.

This provides a recursive approach to solving for the Pareto frontier of the set of PPE payoffs, but not a very convenient or tractable one. The following section develops an alternative procedure to characterize the Pareto frontier that relies on the differentiable structure of the game. [13]

5 A Differential Approach

As in APS, an algorithm of successive approximations can be defined by iterating on the operator B_T, starting from a set containing V^* (such as the set of feasible and individually rational payoffs defined above). This procedure converges monotonically (by set inclusion) to V^*. The algorithm can be simplified in our case, restricting it to iterations on a value function defined by the frontier of values. It can be shown that starting from the frontier of the set of feasible

[13] A related procedure was developed by [21] for a continuous time game with stochastic diffusion processes.

and individually rational payoffs, convergence to the frontier of V^* is monotonic.

Recent work by Sannikov [21] for Brownian games with moral hazard provides a direct method to solve for the boundary of the equilibrium set, given by an ordinary differential equation. In this section we develop an analogue to that result for our Poisson game, and show that the boundary $B(v)$ of the equilibrium set V^* solves an integrodifferential equation. Using this characterization we provide additional results on the equilibrium behavior.

5.1 Recursive Formulation

The derivation of continuous time Bellman equations for the Poisson case is well known.[14] We follow here a heuristic approach. Note that

$$W(v(0)) \geq \alpha \int_0^T \beta^t \left\{x_1(t)\, b - x_2(t)\, c + W(v_1(t)) + W(v_2(t))\right\} dt + \beta^T W(v(T))$$

for any feasible path $\{x_i(t), v_i(t), w_i(t), v(T)\}$, with equality at the optimal path. For constant paths $\{x_i, v_i, w_i, v(T)\}$ and subtracting $W(v(0))$ on both sides gives[15]:

$$\begin{aligned}\alpha \frac{\left(1-\beta^T\right)}{r+2\alpha}&\left[x_1 b - x_2 c + W(v_1) + W(v_2)\right] \\ &+\beta^T\left[W(v(T)) - W(v(0))\right] - \left(1-\beta^T\right) W(v(0)) \leq 0\end{aligned}$$

with equality only at the optimal choices. Dividing through by T and taking limits and letting $v(0) = v$ gives:

$$\alpha\left\{x_1 b - x_2 c + W(v_1) + W(v_2)\right\} + W'(v)\dot{v} \leq (r+2\alpha)\, W(v)$$

with equality at the optimal plan $\{x_i, v_i, w_i, \dot{v}\}$, where following a similar procedure in equation (4.1) gives:

$$\dot{v} = -\alpha\left\{x_2 b - x_1 c + v_1 + v_2\right\} + (r+2\alpha)\, v$$

[14] As a reference, see [?].

[15] Though constant paths for lengths of period T are usually not optimal, this constraint disappears as T converges to zero. This is the analogue of approximating an arbitrary policy function with a sequence of step functions.

Finally, the incentive constraints read:

$$\begin{aligned} v_1 - x_1 c &\geq v \\ W(v_2) - x_2 c &\geq W(v). \end{aligned}$$

In this continuous time problem, the choice variables are x_1, x_2, v_1, v_2 and $\dot{v}$, where the first four variables are the analogue of the controls $x_1(t)$, $x_2(t)$, $v_1(t)$, $v_2(t)$ in the previous problem, and $\dot{v}$ is a drift term and is the analogue of choosing $v(T)$. The optimization problem can be rewritten as:

$$(r+2\alpha)W(v) = \max_{x_1,x_2,v_1,v_2,\dot{v}} \alpha(x_1 b - x_2 c + W(v_1) + W(v_2)) + W'(v)\dot{v} \quad (5.1)$$

subject to:

$$rv = \alpha(x_2 b - x_1 c + v_1 - v + v_2 - v) + \dot{v} \quad (5.2)$$

$$v \leq v_1 - x_1 c \quad (5.3)$$

$$W(v) \leq W(v_2) - x_2 c. \quad (5.4)$$

The following properties can be established as a result of the concavity of the function W.

Proposition 5. *There is an optimal solution to the optimization problem defined by (5.1)-(5.4) with the following properties:*

1. Both incentive constraints bind.
2. $x_1 = \min((v_h - v)/c, 1); x_2 = \min((W(0) - W(v))/c, 1)$.

If $W'(v) \neq W'(v+c)$ the conditions are necessary for player one. Likewise, if $W'(v) \neq W'(v - \delta c)$ where $\delta = 1/W'(v)$, the conditions are necessary for player two.

Proof. See appendix. □

As in the SCM, favors are done while the players' values are away from the boundary. Full favors are granted unless there is not enough utility in the set to compensate the provider for for the cost of the favor. In that case, the size of the favor is limited by the distance to the boundary (divided by the unit cost).

5.1.1 Algorithm

Proposition 5 suggests an algorithm to solve for the value function. This algorithm provides a mapping from the space of a.e. differentiable functions into itself. Start with a candidate value function $W(\cdot)$ and define a new value function $TW(v)$ as follows:

1. Set x_1 and x_2 according to the above Proposition.
2. Let $v_1 = v + x_1 c$ and define v_2 defined implicitly by $W(v_2) - W(v) = x_2 c$.
3. Obtain $\dot{v}$ as a residual from (5.2)
4. Define $TW(v)$ as follows:

$$rTW(v) = \alpha \left(x_1 b - x_2 c + W(v_2) - W(v) + W(v_1) - W(v) \right) + W'(v)\,\dot{v}$$

It should be emphasized that this method is not a contraction mapping. However, the procedure was used to compute the results presented in section 8 and exhibited very fast convergence properties.

5.2 Folk theorem

As indicated above, the SCM mechanism is assymptotically efficient in the sense that as $r/\alpha \to 0$, the expected times at corners goes to zero. As the SCM frontier is included in V^*,the boundary of V^* must converge to the boundary of the set of feasible and individually rational payoffs. More precisely, take the point (v_0, v_0) in the boundary of V^* where $W(v_0) = v_0$. This point corresponds to the highest symmetric payoffs (and by concavity also the highest total payoffs.) As in the SCM mechanism, as $r/\alpha \to 0$ it must be the case that $\frac{r}{\alpha} v_0 \to (b-c)$, the point in the Pareto frontier that corresponds to players giving favors at all opportunities. By Proposition 3, the slope of the boundary of V^* to the left (right) of (v_0, v_0) is lower steeper (flatter) than the Pareto frontier. This implies that as $\frac{r}{\alpha} v_0$ converges to this kink point of the Pareto frontier, the whole boundary $\frac{r}{\alpha} W(v)$ converges to the Pareto frontier. (This can be easily seen from figure 4.1, where it is obvious that the *ray* distance from the point (v_0, v_0) to $(b-c)$ is larger than the ray distance for any other point in the boundary of V^*.). This proves:

Proposition 6. *As $\frac{r}{\alpha} \to 0$, $\frac{r}{\alpha} W(v)$ converges pointwise to the Pareto frontier.*

5.3 Connections to theory

Our game can be reformulated as a more standard repeated game with moral hazard and imperfect monitoring. Each agent has two actions available, $\{n, f\}$ representing *no favor* and *favor,* respectively. If an agent chooses action f, payoffs $(-c, b)$ occur with Poisson arrival α. Information arrives concurrently with favors in the obvious way. This game is in the class considered in [22], that studies the optimal use of information for incentive provision in Public Perfect equilibria. In their setting, actions are fixed for an interval of time - as in our heuristic analysis above. The paper establishes that as this holding time goes to zero, points in the frontier of PPE can be arbitrarily obtained partitioning information in each period into three points: $\{0, 1, 2\}$ corresponding repectively to *no arrivals, first arrival by player one, first arrival by player two.* Though they do not consider equilibria in the continuous time game, their results relate very closely to the formulation given by (5.1)-(5.4), where the *drift* $\dot{V}$ corresponds to the change in value if no favor occurs, while V_1 and V_2 represent continuation values in the other two cases. Section 11 considers a relaxed problem and an outer approximation to the equilibrium set V^* that has some parallels to [22].

There is an alternative way of formulating our model. The actions available to players and the signal structure is the same as above. Payoffs are given by flows $\frac{1}{\alpha}\{(b-c, b-c), (b, -c), (-c, b), (0, 0)\}$ corresponding to strategies $\{(f, f), (n, f), (f, n), (n, n)\}$, respectively. To preserve the parallel to our model, Public Perfect equilibria is conditional on the public signals and not the actual payoffs of the agents. In case of fractional favors, interpret the costs and benefits proportional to the size of the favors. This setting gives rise to the same functional equations and optimization problem given by (5.1)-(5.4). This representation of our game is considered in [12]. Note that the signal structure of this game is similar to the *good news* case in [2]. In contrast to their paper where efficiency is not achieved as the frequency of moves increases, we proved a Folk theorem above. There are two main differences with respect to their game. First, they consider strongly symmetric equilibria where after any history public history continuation values are the same for both players. Secondly, we have a binary signal structure related to each agent's actions.

A general Folk theorem for discrete time games with private monitoring is considered in [9]. Though our continuous time model does not directly map into their framework, the discrete time version considered in [22] does. It is also interesting to note that the proof in [9] relies on supporting payoffs with hyperplanes that are tangent to the frontier of the set of equilibrium values, which as the rate of discount goes to zero correspond to continuation payoffs arbitrarily close to the frontier. [5] analyze a case where the Pareto frontier has an interval with slope -1 around the equal treatment point in a game where private information has only two possible states. Moreover, they provide conditions that make this interval self-generating.

Our results on self-generation is different to [9], as it holds with discounting. It also differs substantially from [5] in several dimensions: 1) our Pareto frontier has no segments with slope -1 (it has the pie shape as seen in Figure 4.1); 2) Our equilibrium set is in the interior of this set; 3) we do not rely on supporting payoffs with hyperplanes and 4) the entire frontier of the equilibrium set of values is self-generating and not only a subset. As far as we know, our self-generation result and the intuition we provide in Section 4.3 are new to the literature.

A recent paper [1] considers a restricted set of equilibria in a game that is related to ours. Their construct parallels [5] by relying on self-generating lines in the space V^*. This is a restriction that we obviously do not impose and is not satisfied by simple mechanisms such as the chips mechanism discussed above.

Finally, our model is obviously less general than some of those considered in this literature. In exchange for that, we are able to characterize with much detail the equilibrium strategies and set of values and provide new insights, as those examined in the next section.

6 Optimal exchange and forgiveness

This section discusses properties of the extremal equilibria. Throughout our discussion we assume incentive constraints bind, which as we argue in Proposition 5 is without loss of generality and necessary in regions where the slope of the boundary changes.[16] As indicated above, the SCM scheme has the property that *prices* of favors are fixed and there is no change in values in absence of favors. This is not the case for extremal equilibria.

Consider first the price of exchanges. Suppose agent one receives a favor. How many favors would she have to give in return to reestablish the original continuation utilities? This favor implies going from v to v_2, where $W(v_2) - W(v) = c$. We can estimate $v_2 \simeq v + c/W'(v)$. The number of favors that agent one needs to make to get back to v is then approximately $\frac{v-v_2}{c} = \frac{1}{W'(v)}$ so we can think of this as the price of a favor. By concavity, as v decreases the price of this favor increases: the more favors an agent has received, the more costly it is to get a new one. Given the bounds obtained in Proposition 4, $c/b \leq p \leq b/c$. As the numerical results in section 8 indicate, these bounds are very closely achieved in our numerical results.

Our second set of results relate to $\dot{v}$, which is arbitrarily set to zero in the SCM scheme. From equation (5.2) and binding incentive constraints it follows that:

$$\dot{v} = rv - \alpha x_2 b - \alpha(v_2 - v) \tag{6.1}$$

where $W(v_2) - W(v) = x_2 c$. It immediately follows that in the extremes $\{0, v_h\}$, as $x_2(x_1) = 0, \dot{v} = 0$. The same must be true at the the symmetric point in the boundary v_0, where $v_0 = W(v_0)$. It is apparent that in general the sign of $\dot{v}$ will depend on the curvature of W and we have not been able to establish general results. However, using the folk-theorem result we can sign $\dot{v}$ as $\frac{r}{\alpha} \to 0$ as the boundary converge to the first best Pareto lines. To the left of v_0, the slope is $-c/b$ and $v_2 - v = c/W'(v) = -x_2 b$. Substituting in (6.1) gives $\dot{v} = rv > 0$. By symmetry, to the right of the equal treatment point v_0 it follows that $\dot{v} < 0$ so that v drifts towards the *equal treatment* point v_0: agents' values are constantly changing over time even when no favors are observed, and the player with a lower entitlement is gradually rewarded. This is what we call forgiveness.[17]

[16] In particular, this would happen if $W(v)$ is strictly concave, as suggested by our computations.

[17] Though this result holds asymptotically as $\frac{r}{\alpha} \to 0$, in all our simulations the same property was verified.

7 On the accounting of favors

In the equilibrium described above, the player's values provide an accounting device of past history and current entitlements. An alternative equivalent accounting is given below. Let T_i represent the expected discounted number of favors that player i will give in the rest of the game. It obviously follows that:

$$v = T_2 b - T_1 c$$
$$w = T_1 b - T_2 c$$

which solving gives:

$$T_1 = \frac{vc + wb}{b^2 - c^2} \tag{7.1}$$
$$T_2 = \frac{wc + vb}{b^2 - c^2} \tag{7.2}$$

and

$$T_1 - T_2 = \frac{w - v}{b + c}.$$

This difference can be understood as a net balance between the player's assets and liabilities. When $T_1 < T_2$, the player is in a net debt position and as we found, $\dot{v} > 0$ in this region. The process T_1 and T_2 are useful to think about the way the optimal mechanism grants values and provides incentives. As an example, in the SCM scheme T_1 and T_2 are related to the allocation of chips. As the chips change hands, both T_1 and T_2 change: a player is rewarded after giving a favor both, by increasing the entitlement and decreasing the obligation to grant future favors.

It is instructive to consider this accounting on the Pareto frontier. At the equal treatment point (v_0, v_0), $T_1 = T_2 = \frac{1}{1-\beta}$. To the left of this point, $(w - w^*) / (v - v_0) = -c/b$ so T_2 is constant and all the adjustment is obtained by lowering T_1. In the boundary of our set of PPE payoffs V^*, for values of v where $\frac{c}{b} < \|W'(v)\| < \frac{b}{c}$ incentives are provided both by changes in T_1 and T_2 in similar fashion to the chips mechanism. More precisely, using (7.1) and (7.2) on the frontier $W(v)$ it is easy to verify that T_2 (resp. T_1)will be decreasing in v (decreasing in w) when $W'(v) < -c/b$(resp. $W'(v) > -b/c$). We can prove (see Lemma 8), that this indeed occurs for $v > c(resp.w > c)$, so for most of the domain -if not all- the optimal mechanism rewards a player's favors with an increase in its own entitlement to the other player's favors *as well as* a reduction in the entitlements of the other player.

8 Numerical Results

The optimal strategies differ in several dimensions from the very simple strategies proposed by the SCM of Mobius. How important is this? This section provides some numerical computations to examine this question. There are 4 parameters in the model: r, α, c, b. In comparing the performance of different alternatives, two normalizations can be made where all that matters in these comparisons are the values of c/b and r/α. In the next tables, the following normalizations are used: $b = 1$ and $r = 0.01$.

Table 1 gives a measure of how far each alternative scheme is from the first best at the symmetric point of the boundary where players get equal utilities. The first column gives the percentage difference between the first best values and values for the optimal scheme described above; the second column gives the difference with SCM values.

Table 1: Efficiency of PPE and SCM

% difference with optimum					
$\alpha = 1$			$c = 0.5$		
c	PPE	SCM	α	PPE	SCM
0.25	1.3	4.0	0.1	9.0	23.8
0.5	2.7	7.2	0.25	5.5	13.2
0.75	3.5	12.2	1	2.7	7.2
0.95	9.8	33.2	10	0.8	2.2

The performance of both these schemes decreases with c and increases with the arrival rate α. There can be substantial improvements over the SCM: e.g. for $\alpha = 1$ and $c = 0.95$, the second best is 10% within the first best, while the SCM is over 30% apart.

Table 2 gives the maximum number of consecutive favors that can be supported with both schemes. For the optimal PPE equilibria this is calculated as $\bar{v}/c$

while in the SCM mechanism it is the maximum number of chips supported. The optimal scheme can accommodate a much larger number of favors (between 5 to over 10 times more). In part this is due to flexible relative prices, which depart significantly from one. A natural question is, when do fixed prices become a good enough substitute for flexible prices? Our computations show that, as expected, when trade opportunities are very frequent (or equivalently, when dicounting is very low), fixed prices approach in their performance flexible prices. The table also shows the degree of inequality that can be generated from each of these schemes, as measured by the percentage difference between the point of equal treatment $v^*$and the maximum attainable value. While inequality is quite significant in the optimal PPE it is fairly limited for the SCM. As suggested by Proposition 1, inequality tends to vanish as $r/\alpha \to 0$. As an example, when $c = 0.5$ and $\alpha = 10$, the optimal SCM supports a degree of inequality slightly over the first best value of 50%, the corresponding value for the SCM is just 2.2%.

Table 2: Favors, prices and inequality

		Number of favors		*Average price*	$\bar{v}/v^* - 1$		
c	α	PPE	SCM	*left of* v^*	PPE	SCM	first best
0.25	1	375	21	3.75	26.7%	4.1%	25%
0.5	1	150	13	1.85	54.1%	7.0%	50%
0.75	1	58	8	1.23	81.1%	12.3%	75%
0.95	1	9.4	3	1.02	97.7%	28.6%	95%
0.5	0. 1	15	4	1.57	63.9%	22.2%	50%
0.5	0.25	37.5	7	1.70	58.7%	14.7%	50%
0.5	1	150	13	1.85	54.1%	7.0%	50%
0.5	10	1500	42	1.95	51.2%	2.2%	50%

In all the simulations, $\dot{v}$ is positive for values of v under the symmetric point v_0 so the equilibrium displays forgiveness. Figure (8.1) illustrates this for the benchmark case. It is interesting to note that $\dot{v}/v$ is monotonically decreasing and that it almost equals the interest rate $r = 1\%$ in the lower section of its domain. Using (5.2) and the incentive constraint for player one it follows that:

$$r - \frac{\alpha}{v}\left(x_2 b + v_2 - v\right) = \frac{\dot{v}}{v}$$

so that $v_2 - v \approx x_2 b$. Using the incentive constraint for player two this implies that in this range of values $W'(v)$ must be approximately equal to $-c/b$ so that player one is almost indifferent between receiving an extra favor or not.

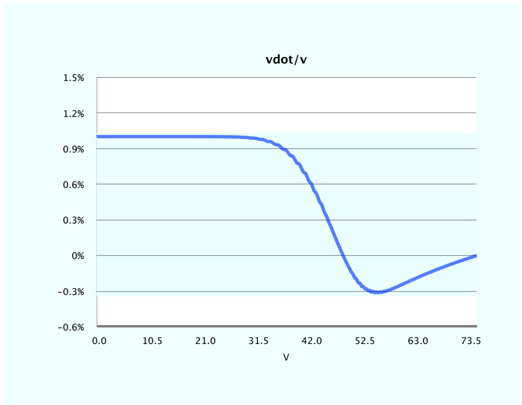

Figure 8.1: Forgiveness

Figure (8.2) provides the decomposition of values in terms of *favor entitlements* indicated above. Note that for lower values of v most of the increase in value for player one is the result of an increased entitlement to favors of player two (T_2), with basically no change in the favors owed by player one.

Total value $(v + w)$ is maximized at the point of equal treatment. This follows formally from concavity and symmetry and intuitively from the fact that moving towards the endpoints of the domain increases the probability of hitting a corner where one of the players will not receive favors. This can be seen very clearly in Figure 8.3 that represents the total discounted favors $(T_1 + T_2)$ as a function of v. Note the wide gap, where at the endpoints the total is 150 while at the point v^* of equal treatment the value is closer to 195 (as a reference, in the first best this would be 200.)

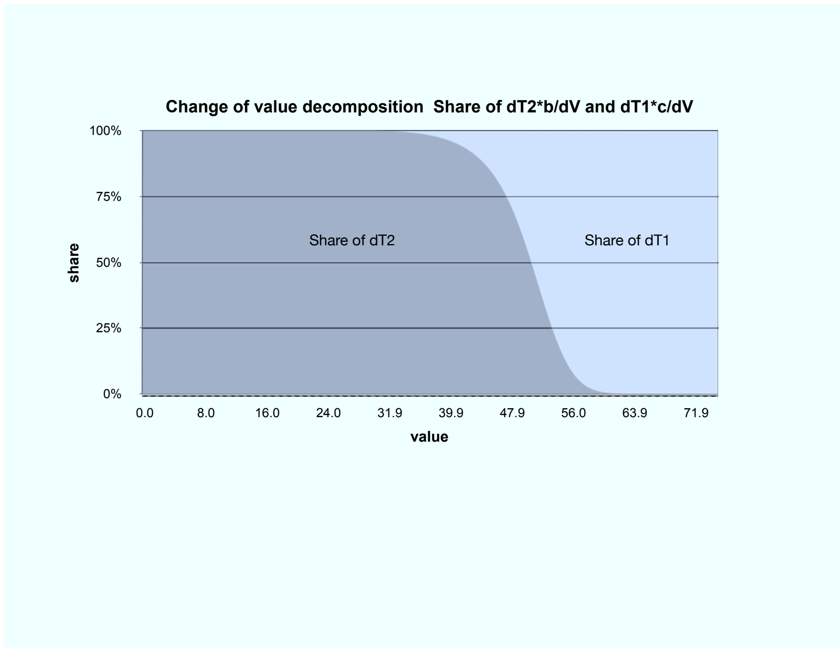

Figure 8.2: Decomposition of value change

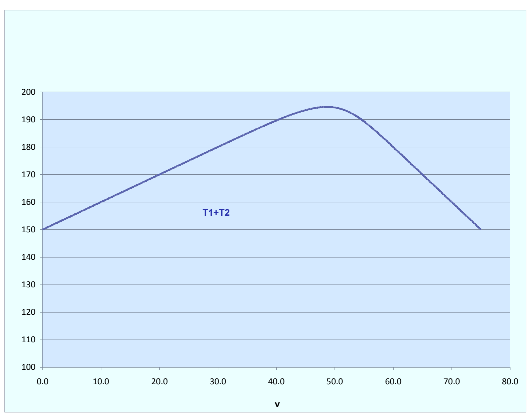

Figure 8.3: Total Entitlements to Favors

9 A generalized Chips Mechanism: Forgiveness and Inflation

A number of papers in macroeconomics have addressed the potential optimality of expansionary monetary policy ([13], [20], [7], [8], among others) in economies where there is lack of coincidence of needs and private information. In a nutshell, inflation can act as an insurance device which tends to re-balance agents' wealth in favor of the least "lucky", and the purpose is to avoid hitting extreme distributions which could lead to inefficiencies. The two most relevant papers for us are by [11] and [18].

[11]study an economy where agents' preferences and endowments are private information and random. Every period, individuals make donations to a common fund, and the proceeds are redistributed over them by a social planner. An agent's wealth depends directly on the past donations and received transfers of that agent and represent a future claim on the common fund, thus acting as insurance against fluctuations in endowment. The authors show that, in the case where utility functions are linear, an expansionary monetary mechanism Pareto dominates the mechanisms with constant or contractionary monetary policy. The reason is that inefficiencies here occur when low marginal utility agents consume their endowment instead of contributing to the common fund. By constantly inflating money balances, agents' wealth remain bounded from above, so they never become wealthy enough to fully insure themselves against future fluctuations in their endowment.

[18]gives an example with two types of infinitely lived agents that shift randomly between having high and low valuation of consumption, and where the type is private information. He shows that if the difference in valuation is large enough, there will be an expansionary policy that dominates all contractionary policies. The intuition is that a sufficiently long sequence of realizations where a group is the high type will result in that group running out of money, hence not affording to consume any longer when their valuation is high. In that case, an expansionary policy (say, through lump sum transfers to all agents in the economy) will tend to insure unlucky buyers.

The difference between the two cases is subtle, but Levine's argument is actually analogous to the rationale for having $\dot{V}$ in our mechanism. One can think of agents valuing a favor differently whether they are givers or receivers, and the giver is always compensated with higher future claims on favors. Since the Individual Rationality constraint puts the bound on the exchange, there is no risk of an agent keeping a favor to herself, but rather, that her partner hits the zero value limit and has nothing to trade for a favor. $\dot{V}$, by constantly working against the agent with a higher value, has a similar function to an inflationary policy which partially insures agents against reaching that point.e. the larger domain of values and the variable relative prices of favors, can potentially accommodate a considerably larger number of favors before reaching the boundaries.

I constrast to our model, these papers consider an environment with a large number of agents. In order to draw a parallel to this literature, this section examines implementation of our equilibrium with a *chips mechanism.* We first define such a mechanism and provide a mapping from values to *chips.* We then discuss the connection between forgiveness and *inflation* generated by the injection of chips.

Definition 7. A chips mechanism induced by the equilibrium of the game is a strictly increasing mapping $c(v)$ from agent one's value to the interval $[0, 1]$ with the symmetry property that: $c(v) = 1 - c(W(v))$.

The value $c(v)$ is interpreted as the share of chips held by agent one. The first property says it is an accounting device sufficient for determining the values of the players. The second property implies that the share of chips fully determines the utility of a player independently of its identity. These are two conditions satisfied by Mobius' scheme.

As an example, let $c(v) = v/(v + W(v))$. It is easy to verify that both properties of the definition are satisfied. In our equilibrium prices depend on the shares of chips as follows:

1. Player one does a favor: $p_1(c) = c(v_1) - c(v)$ where $c = c(v)$.
2. Player two does a favor: $p_2(c) = c(v) - c(v_2)$ where $c = c(v)$.

The share of chips also changes independently of favors over time: $\dot{c}(c) = c'(v)\dot{v}(v)$ where $c = c(v)$. Subject to this *forgiveness rule*, the equilibrium can be implemented by a chips mechanism where the receiver of a favor has all the bargaining power to determine the price (in terms of chips) of a transaction. This follows immediately from the fact that in our equilibrium the incentive compatibility constraint binds for the agent doing the favor.[18]

Forgiveness suggests negative real interest rates. These can be obtained by a specific injection of chips. Let $m(t)$ denotes the total number of chips at time t and $m_i(t)$ the number held by player i. Then $m_1(t)/m(t) = c(t)$. Let $\dot{m}(t)$ denote the injection of chips at time t and suppose that $\dot{m}_1(t) = \dot{m}_2(t) = \dot{m}(t)/2$ so that both players receive the same number of chips. All prices grow at the same rate as the stock of chips. Note that

$$\begin{aligned}\dot{c}(t) &= \frac{d}{dt}\frac{m_1(t)}{m(t)} = \frac{m(t)\,\dot{m}(t)/2 - m_1(t)\,\dot{m}(t)}{m(t)^2} \\ &= \frac{\dot{m}(t)}{m(t)}\left(\frac{1}{2} - c(t)\right)\end{aligned}$$

so the rate of expansion of chips

$$\frac{\dot{m}(t)}{m(t)} = \frac{\dot{c}(t)}{\frac{1}{2} - c(t)}, \tag{9.1}$$

which is positive whenever the sign of $\dot{c}(t)$ equals the sign of $\frac{1}{2} - c(t)$, as occurs in our computations.

Relation of $\dot{v}$ to monetary inflation:

An important caveat to this analysis, is that the computation given by (9.1) is units-dependent. Any monotone transformation of a generalized chips mechanism that preserves symmetry is also a chips mechanism and will give a different rate of inflation. There is one exception, and this is the limit inflation rate as $V \to V_0$, the equal treatment point. At this point $c(t) = \frac{1}{2}$ and $\dot{c}(t) = c'(V_0)\,\dot{V} = 0$ as $\dot{V} = 0$ at this point. Using L'Hopital's rule:

$$lim_{V \to V_0} \frac{c'(V_0)\,\dot{V}(V_0)}{\frac{1}{2} - c(V_0)} = lim_{V \to V_0} \frac{c''(V_0)\,\dot{V}(V_0) + C'(V_0)\,\dot{V}'(V_0)}{-C'(V_0)} = -\dot{V}'(V_0),$$

[18]On the contrary, if all the bargaining power were on the sellers side, the buyer would be made indifferent. But in that case, a seller would never sell for it would not be able to realize any future gains from this trade. This was pointed out to us by Andy Skrzypacz.

independently of the representation c. From our computations, we also found this to be the peak in terms of rate of inflation. As an example, in our baseline model with arrival rate $\alpha = 1$, this number is 4.4% while for $\alpha = 0.1$ it is 0.9% and for $\alpha = 10$ it is 7%. Figure 9.1 gives the implied rates of chips expansion (i.e. inflation) implied by the equilibrium levels of forgiveness in our baseline case for the representation given above. The peak occurs at the symmetric point reaching a value of approximately 4% (four times larger than the discount rate $r = 1\%$.). Since the rate of inflation is equivalent to the negative interest rate implied, an alternative implementation is to have approximately 4% constant rate of expansion of chips and charge an interest to fill the gaps which according to the figure would increase with the size of the debt $\left(\frac{1}{2} - c\right)$, from zero to approximately 4% as it approaches its maximum at the extremes of the interval.

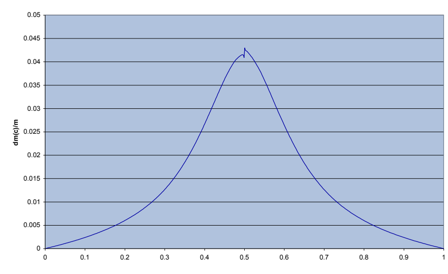

Figure 9.1: Optimal rates of chips expansion

10 Final remarks

This paper considers cooperation in the absence of immediate reciprocity in a partnership. The lack of double coincidence of needs has been the subject of many papers in monetary theory. In most of that literature, trading requires the existence of a medium of exchange, usually money, or chips. Money is essentially a record keeping device which can be seen as a "primitive form of memory" [16], and is especially practical when players are numerous and do not act repeatedly. In our paper the repeated interaction of players makes exchange possible as an equilibrium outcome though informational frictions put a limit on what can be achieved. As seen in the computations, this cost decreases significantly with the frequency of trading opportunities α.

In many organizations internal exchange is not mediated with money. Obvious examples are households and other partnerships such as co-authors or coworkers. Our analysis suggests that the lack of use of money should be related to the frequency of trade opportunities. Casual observation suggests that in many of these organizations, the multiple dimensions of exchanges enhance the frequency of trading opportunities, thus reducing the value of mediating trade with monetary payments.

Our results highlight two important features of the optimal mechanism. One is the obvious need for inequality to sustain the optimal arrangements. Because of the lack of coincidence, current trades add a liability in terms of future favors to the receiver. As this liability increases with the number of favors received, to sustain trade the rate of exchange of future for present favors -its relative *price*- increases. One particular implication of this rise in relative prices is that to reestablish a position of equal treatment, a player that received a series of favors must retribute the other player with even more net favors. This suggests a more general principle, that it can be quite costly to recover from a *bad reputation,* in our case the reputation of *receiven but not giving enough*. Our mechanism leads to considerably more inequality than a simple chips mechanism, that is one with a medium of exchange and fixed relative prices. In contraposition, the optimal mechanism is *forgiving*, that is a drift in continuation values to the equal treatment point. Our results show that these two features can enhance significantly the performance relative to the simple chips mechanism.

Equilibrium values in our model are associated with entitlements to receive and obligations to grant future favors that can be represented as expected discounted favors. This is true for any equilibrium, whether the standard chips mechanism or our optimal one. In the first case, where trade is mediated by a standard medium of account, a player is compensated after granting a favor both by an increase in the entitlement to other favors as well as a reduction in the obligation to grant future favors. This is obviously true by construction: as one player gets a larger share of chips. the other players corner situation of *no chips* is expected to occur sooner while the opposite is true for the given player. Our optimal PPE has also this property, but the mix of decreased liabilities and increased entitlements in rewarding a player change significantly with the entitlements: while the latter dominates for low values, the former is more important for high values. In this respect, our mechanism is in between the standard chips mechanism and the first best, where transfers of value between players occur only with unilateral changes in the favors entitled to the disadvantaged player.

Our paper is related to a growing literature on dynamic incentives. Though are model is more specialized in the nature of payoffs, it analyzes a very relevant case not considered before: the problem of *dynamic insurance* under *private information* in *small groups*. Moreover, our continuous time formulation seems to provide considerable methodological advantages, such as self-enforcement of the Pareto frontier and the associated integro-differential formulation. There are obvious interesting extensions, such as the case of strictly concave preferences and the characterization of equilibria with more players.

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Appendix A: Proofs of Propositions:

Lemma 8. *The boundary $W(v)$ satisfies the following properties.*

1. The function $W(v) < O(v)$ for all $0 < v \leq v^*$ where $O(v) \equiv \frac{\alpha}{r} min\{\left(b - \frac{c^2}{b}\right) - \frac{c}{b}v$ is the corresponding point in the Pareto frontier and $W(v^*) = v^*$.
2. For any $v > c,\ W'(v) < -c/b$.

Proof. To prove the first part, suppose towards a contradiction that there exists a point $0 < v_0 \leq v^*$ where $W(v_0) = O(v_0)$. Without loss of generality, assume this is the largest value v for which this equality holds. Since the slope of $W(v) \leq -c/b$ and $O(v) \geq W(v)$, this implies that $W(v) = O(v)$ for all $v \leq v_0$ and $W'(v) = -c/b$. Using (5.2) and binding incentive constraint (5.3), it follows that:

$$v_0 = rv_0 - \alpha x_2 b - \alpha (v_2 - v_0). \tag{10.1}$$

Furthermore, using the incentive constraint for player two, $W(v_2) - W(v) = x_2 c$, which by the slope of W implies $v_2 - v = -b$. Substituting in (10.1) this implies that $\dot{v} = rv$. Now consider the value for player one, assuming his incentive constraint binds:

$$\begin{aligned} rW(v_0) &= \alpha b + \alpha (W(v_0 + c) - W(v_0)) + W'(v_0)\dot{v} \\ &= \alpha b + \alpha (W(v_0 + c) - W(v_0)) - \frac{c}{b} r v_0. \end{aligned} \tag{10.2}$$

(We assume that W is differentiable at v_0, but if not we can take instead a point arbitrarily close to v_0 at the left and consider the left derivative.)

Substituting $W(v_0) = \frac{\alpha}{r}\left(b - \frac{c^2}{b}\right) - \frac{c}{b}v_0$ in the left hand side gives:

$$\alpha\left(b - \frac{c^2}{b}\right) - \frac{c}{b} r v_0 = \alpha b + \alpha (W(v_0 + c) - W(v_0)) - \frac{c}{b} r v_0 \tag{10.3}$$

which after simplifying gives $W(v_0 + c) = W(v_0) - \left(\frac{c}{b}\right)c \equiv O(v_0 + c)$, contradicting the fact that v_0 was the highest value for which $W(v) = W(v_0)$ so it must be that $v_0 = v^*$. But this contradicts the fact that $W(v^*) < O(v^*) = \frac{r}{\alpha}(b - c)$ since this value can only be achieved with both players doing always favors, which is not incentive compatible.

To prove the second part, take $0 < v_0 < c$ and suppose that $W'(v_0) = -c/b$ (otherwise, the proof is complete.) By the first part, $rW(v_0) < \alpha\left(b - \frac{c^2}{b}\right) - \frac{c}{b}rv_0$ so instead of equation (10.3) the following inequality is obtained:

$$\alpha\left(b - \frac{c^2}{b}\right) - \frac{c}{b}rv_0 > \alpha b + \alpha\left(W(v_0 + c) - W(v_0)\right) - \frac{c}{b}rv_0$$

implying that $\frac{W(v_0+c)-W(v_0)}{c} < -\frac{c}{b}$. Since this is true for v_0 arbitrarily close to zero, the claim is proved. □

Lemma 9. *Suppose that* $\frac{\alpha}{r}(b - c) > c$. *Then the set* V^* *has nonempty interior. Moreover, in that case* $W(0) > 0$.

Proof. Consider a single chip mechanism. Only the player with no chip does a favor. It is easy to show that $v_1 - v_0 = \frac{\alpha(b+c)}{r+2\alpha}$ and this will exceed c if and only if the above inequality holds. This strategy gives a strictly positive value for both players (v_0, v_1) to both players. Moreover, by Proposition 8, extending this point to the left on slope $-c/b$ gives the point $(0, v_1 + v_0c/b)$. This proves the second part of the lemma. Furthermore, since $V^*$is convex, it contains the convex hull of these two points and the origin and thus has non-empty interior. □

Proof of Proposition 4

Take a point (v, w) in the Pareto frontier of V^*. For any T, this can be factorized by strategies and continuation values $\{x_i(t), v_i(t), w_i(t), v_T, w_T\}$. Suppose (v_T, w_T) is not in the Pareto frontier, so there exists $\varepsilon > 0$ such that $(v_T + \varepsilon, w_T + \varepsilon) \in V^*$. Define new paths $\tilde{x}_i(t) = \max(0, x_i(t) - d(t))$, where $d(t) = \frac{\varepsilon}{c}e^{\alpha(T-t)}\beta^{T-t}$. We prove that these paths together with

$$\{v_i(t), w_i(t), v_T + \varepsilon, w_T + \varepsilon\}$$

are admissible with respect to V. First we show incentive compatibility. Letting

$$z_1(s) = x_2(s)b - x_1(s)c + v_1(s) + v_2(s),$$

$$
\begin{aligned}
&\alpha \int_t^T \beta^{s-t} \left\{ \tilde{x}_2(s) b - \tilde{x}_1(s) c + v_1(s) + v_2(s) \right\} ds + \beta^{T-t} (v_T + \varepsilon) \\
&\leq \alpha \int_t^T \beta^{s-t} \left\{ x_2(s) b - x_1(s) c + v_1(s) + v_2(s) + d(s)(c-b) \right\} ds + \beta^{T-t} (v_T + \varepsilon) \\
&\leq \alpha \int_t^T \beta^{s-t} \left\{ x_2(s) b - x_1(s) c + v_1(s) + v_2(s) \right\} ds + \beta^{T-t} (v_T + \varepsilon) + \alpha c \int_t^T \beta^{s-t} d(s) ds \\
&= \alpha \int_t^T \beta^{s-t} z_1(s) ds + \beta^{T-t} (v_T + \varepsilon) + \alpha \varepsilon \beta^{T-t} \left(\frac{-1 + e^{\alpha(T-t)}}{\alpha} \right) \\
&= \alpha \int_t^T \beta^{s-t} z_1(s) ds + \beta^{T-t} v_T + d(t) c \\
&\leq v_1(t) - x_1(t) c + d(t) c.
\end{aligned}
$$

For $\tilde{x}(t) > (d(t) < x(t))$,

$$
v_1(t) - x_1(t) c + d(t) c = v_1(t) - \tilde{x}_1(t) c
$$

so the incentive constraint for player one is satisfied. In contrast, if $\tilde{x}(t) = 0$ the incentive constraint for player one is not relevant anymore.[19] A similar argument shows that the same holds for player two. Let $(\tilde{v}, \tilde{w})$ denote the values associated to the new admissible path. Then

$$
\begin{aligned}
\tilde{v} &\geq \alpha \int_0^T \beta^t z(s) ds + \beta^T (v_T + \varepsilon) + \alpha(c-b) \int_0^T \beta^t b d(t) dt \\
&= v + \beta^T \varepsilon + \beta^T \varepsilon (e^T - 1) \left(1 - \frac{b}{c} \right) \\
&= v + \beta^T \varepsilon \left(e^T - e^T \frac{b}{c} + \frac{b}{c} \right)
\end{aligned}
$$

Picking T sufficiently small (so $e^T \left(\frac{b}{c} - 1 \right) < \frac{b}{c}$),the right hand side exceeds v. A similar argument shows that $\tilde{w} > w$,contradicting the hypothesis that the value pair (v, w) is in the Pareto frontier of V^*. This proves that (v_T, w_T) must belong to the Pareto frontier of V^*.

We now show that $(v_i(t), w_i(t))$ also belong to the frontier. Suppose towards a contradiction that this was not true. Let

$$
a = \min \left\{ v + w \middle| (v, w) \text{ are in the Pareto frontier of } V^* \right\}.
$$

[19]For the purposes of generating the same stochastic process for $\{v_1(t), w_1(t)\}$ the arrival of a favor by player one can be replaced by a randomization device providing a signal with the same arrival rate α.

From the concavity and symmetry of the boundary of V^*, $a = W(0)$ which by lemma 9 is strictly positive. It follows that $v_T + w_T \geq a$. Let T_i be the set of time periods such that $v_i(t), w_i(t)$ are not in the Pareto frontier of V^*. We construct an alternative admissible path that improves on the given one as follows. Let $\hat{v}_i(t)$ be the value such that the pair $(\hat{v}_i(t), w_i(t))$ is in the frontier. Similarly define $\hat{w}_i(t)$. Without loss of generality (by choice of T),

$$\int_0^T \beta^t (\hat{v}_i(t) - v_i(t))\, dt + \int_0^T \beta^t (\hat{w}_i(t) - w_i(t))\, dt < \beta^T a. \tag{A}$$

It is thus possible to construct paths $\{\tilde{v}_i(t), \tilde{w}_i(t)\}$ where for each t and i either $(\tilde{v}_i(t), \tilde{w}_i(t))$ equals $(v_i(t), \hat{w}_i(t))$ or it equals $(\hat{v}_i(t), w_i(t))$ such that

$$\varepsilon_1 = \int_0^T \beta^t [\tilde{v}_1(t) - v_1(t) + \tilde{v}_2(t) - v_2(t)]\, dt \leq \beta^T v_T \tag{B}$$

and

$$\varepsilon_2 = \int_0^T \beta^t [\tilde{w}_1(t) - w_1(t) + \tilde{w}_2(t) - w_2(t)]\, dt \leq \beta^T w_T \tag{C}$$

with strict inequality for at least one of the two. Define $\tilde{v}_T = v_T - \frac{\varepsilon_1}{\beta^T}$ and $\tilde{w}_T = w_T - \frac{\varepsilon_2}{\beta^T}$. We now show that the path $\{x_i(t), \tilde{v}_i(t), \tilde{w}_i(t), \tilde{v}_T, \tilde{w}_T\}$ is admissible. Let

$$\begin{aligned}
\delta_1(t) &= \tilde{v}_1(t) - v_1(t) + \tilde{v}_2(t) - v_2(t) \\
\delta_2(t) &= \tilde{w}_1(t) - w_1(t) + \tilde{w}_2(t) - w_2(t)
\end{aligned}$$

So we have:

$$\begin{aligned}
&\alpha \int_t^T \beta^{s-t} \tilde{z}_1(s)\, ds + \beta^{T-t} \tilde{v}_T \\
&= \alpha \int_t^T \beta^{s-t} z_1(s)\, ds + \beta^{T-t} v_T + \int_t^T \beta^{s-t} \delta_1(s)\, ds - \beta^{-t} \varepsilon_1 \\
&= \alpha \int_t^T \beta^{s-t} z_1(s)\, ds + \beta^{T-t} v_T + \beta^{-t} \left(\int_t^T \beta^{s} \delta_1(s)\, ds - \varepsilon_1 \right) \\
&\leq \alpha \int_t^T \beta^{s-t} z_1(s)\, ds + \beta^{T-t} v_T \\
&\leq v_1(t) - x_1(t)\, c \\
&\leq \tilde{v}_1(t) - x_1(t)\, c.
\end{aligned}$$

A similar argument can be used to verify incentive compatibility for player two. We have constructed paths where $(\tilde{v}_i(t), \tilde{w}_i(t))$ are in the Pareto frontier for all t. To end the proof, note that since $(\tilde{v}_T, \tilde{w}_T)$ are not in the frontier, by our previous argument the path can be further improved by an alternative one that takes values in the Pareto frontier (without modifying $(\tilde{v}_i(t), \tilde{w}_i(t))$.

Proof of Proposition 5

Take the program defined by (5.1)-(5.4). Suppose the incentive constraint for player one does not bind. Consider an alternative vector lowering v_1 by Δv_1 and raising $\dot{v}$ by $\alpha \Delta v_1$ so that the participation constraint (5.2) continues to be satisfied. This leads to a change in (5.1)

$$\begin{aligned}&\alpha\left(W\left(v_1-\Delta v_1\right)-W\left(v_1\right)\right)+W'\left(v\right)\Delta v_1\\&\geq \alpha\left(-W'\left(v_1\right)+W'\left(v\right)\right)\Delta v_1\geq 0,\end{aligned}$$

where the two inequalities follow from concavity and $v_1 > v$. Note that these inequalities will be strict and a binding incentive constraint not only sufficient but necessary, unless the frontier is linear on $[v, v + x_1 c]$. A similar argument applies to the incentive constraint of player two. To show part 2, suppose that $x_1 < \min\left(\left(v_h - v\right)/c\right), 1)$. Assume without loss of generality the incentive constraint binds. Now increase x_1 by Δx_1 and also increase v_1 by $c\Delta x_1$ so that the incentive and participation constraints for player one continue to hold. This gives an immediate increase in utility for player two of $\Delta x_1 b$ and a decrease in future utility $W\left(v_1 + c\Delta x_1\right) - W\left(v_1\right)$ which by Proposition 3 is no greater than $-\left(\frac{c}{b}\right) c\Delta x_1 = -\Delta x_1 b$. The net utility of player two does not fall and provided the bounds in Proposition 3 are strict, it will increase. (By Lemma 8, this will be true in most of the domain, if not everywhere.) A similar argument can be used that $x_2 = \min\left(\frac{W(0)-W(v)}{c}, 1\right)$. Finally note that if $v_h - v \geq c$ and $W'\left(v\right) \neq W'\left(v+c\right)$, our previous argument shows that the incentive constraint must bind. Similar arguments apply to player two.

11 Factorization and Algorithm (not for publication)

(Not for Publication)

In this appendix, we show that values in V^* can be factorized according to the formulation we used without loss of generality, and provide a simple algorithm which approximates the Pareto frontier from above.

Let $B(V)$ denote the APS operator associated to $T = \infty$.

Lemma 10. *If $V \subset B_T(V)$, then $V \subset B(V)$.*

Proof. Let $(v, w) \in B_T(V)$. By monotonicity, (v, w) is also in $B_T^n(V)$ for all n with factorization $(x_i(t), v_i(t), w_i(t), v(nT), w(nT))_{t=0}^{nT}$ and associated values $z_i(t)$ such that:

$$\begin{aligned} v &= \alpha \int_0^{nT} \beta^t z_1(t)\, dt + \beta^{nT} v_{nT} \\ w &= \alpha \int_0^{nT} \beta^t z_2(t)\, dt + \beta^{nT} w_{nT}, \end{aligned}$$

such that as n is increased all previous $z_i(t)$ terms are maintained. Taking the limit of $(x_i(t), v_i(t), w_i(t))$ as $n \to \infty$ delivers a factorization of (v, w) for B. (This is like in a standard dynamic programming problem iterating forward the optimal policy.) □

Lemma 11. *If $V = B(V)$ then $V \subset B_T(V)$.*

Proof. Take $(v, w) \in B(V)$ with factorization $\{x_i(t), v_i(t), w_i(t)\}$ with corresponding values $\{z_i(t)\}$. Let $v_T = \alpha \int \beta^t z_1(t+T)\, dt$ and $w_T = \alpha \int \beta^t z_2(t+T)\, dt$. By definition the associated values $(v_i(t+T), w_i(t+T)) \in V$, so $(v_T, w_T) \in B(V) = V$. It follows immediately that $\{x_i(t), v_i(t), w_i(t), v(T), w(T)\}$ factorizes (v, w) for B_T. □

Corollary 12. *The largest fixed points of B and B_T are the same.*

Proof. Let V be a fixed point of B. By Lemma 11, $V \subset B_T(V)$, so the largest fixed point of B_T contains V. Let V be a fixed point of B_T. By Lemma 10, $V \subset B(V)$, so the largest fixed point of B contains V. $\square$

The outer approximation

We consider a relaxed problem. More precisely, we will say that

$$\left(\{x_i(t), v_i(t), w_i(t)\}_{t=0}^{T}, v_T, w_T\right)$$

is weakly admissible with respect to V if conditions (4.1) and (4.2) hold with the weaker incentive constraints:

$$v_1(t) - x_1(t)c \geq \beta^T v_T$$
$$w_2(t) - x_2(t)c \geq \beta^T w_T$$

Lemma 13. *Suppose that* $\left(\{x_i(t), v_i(t), w_i(t)\}_{t=0}^{T}, v_T, w_T\right)$ *is weakly admissible with respect to convex V. Then the constant paths*

$$(x_1, x_2, v_1, v_2, w_1, w_2, v_T, w_T)$$

defined by:

$$x_i = \frac{1}{\int_0^T \beta^t dt} \int_0^T \beta^t x_i(t)\, dt$$
$$v_i = \frac{1}{\int_0^T \beta^t dt} \int_0^T \beta^t v_i(t)\, dt$$
$$w_i = \frac{1}{\int_0^T \beta^t dt} \int_0^T \beta^t w_i(t)\, dt$$

are weakly admissible with respect to V.

Proof. By convexity of V, the continuation values lie in V and it is also obvious that $0 \leq x_i \leq 1$. So we only need to verify (weak) incentive compatibility. The incentive constraints verify immediately. $\square$

Denote by $\bar{B}_T$ the associated APS operator. Note that by definition, the *averaged* path gives rise to the same values v, w. So without loss of generality in defining $\bar{B}_T$ we can restrict to constant paths. Following the same analysis as in Abreu, Pearce and Stacchetti, this operator has a largest fixed point $\bar{V}_T$ and it is convex and compact. Letting $\bar{V} = \cap_T \bar{V}_T$ it follows that $\bar{V}$ is also compact and convex. To complete the proof, we will show that $\bar{V} = V^*$, the set of PPE.

First note that since $\bar{V}_T$ is defined by weaker incentive constraints, $V^* \subset \bar{B}_T V^*$, i.e. it is self-generating and this implies that $V^* \subset \bar{V}_T$ for all T and thus $V^* \subset \bar{V}$. Intuitively, because the incentive constraints defining $\bar{B}_T$ are weaker than those defining B_T, its largest fixed point contains the set of PPE values. We will now show that $\bar{V} \subset V^*$, which completes the proof.

Let $\bar{V}_n$ be the fixed point of the operator defined according to the weaker incentive constraint $T_n = \frac{1}{2^n}$ and let $\beta_n = exp\left(-\left(r + 2\alpha\right) T_n\right)$.

Preliminaries: constant path factorization We will focus here on a simpler class of strategies that generate values in V^*, where players actions $x_i\left(t\right)$ are constant for an interval $[0, T_n]$ and arbitrary thereafter. Factorization of these strategies requires continuation values $\left(v_1, w_1\right), \left(v_2, w_2\right), \left(v_0, w_0\right)$ all in V^*. Consider the incentive constraint for player one at $t \in [0, T_n]$

$$\begin{aligned} v_1 - x_1 c &\geq \alpha \int_t^{T_n} e^{-(r+2\alpha)s} \left(x_2 b - x_1 c + v_1 + v_2\right) ds + \beta^{T_n - t} v_0 \\ &= \alpha \left(1 - \beta^{T_n - t}\right) \frac{x_2 b - x_1 c + v_1 + v_2}{r + 2\alpha} + \beta^{T_n - t} v_0. \end{aligned}$$

For these constraints to be satisfied for all $t \in [0, T_n]$, it is necessary and sufficient if it holds at the extremes:

$$\begin{aligned} v_1 - x_1 c &\geq v_0 \\ v_1 - x_1 c &\geq v \end{aligned} \tag{11.1}$$

where v is the initial value

$$v = \alpha \left(1 - \beta^{T_n}\right) \left[\frac{x_2 b - x_1 c + v_1 + v_2}{r + 2\alpha}\right] + \beta^{T_n} v_0. \tag{11.2}$$

Similarly for player two

$$w_2 - x_2 c \geq w_0$$
$$w_2 - x_2 c \geq w.$$

Observe that as the term in brackets in (11.2) is bounded so if the first incentive constraint in (11.1) is satisfied with a slack $\varepsilon > 0$ the second constraint will be satisfied automatically for sufficiently high n. We make use of this below.

Lemma 14. *For any* $0 < \delta < 1$, *there exists an* N *such that* $V^* \supset \delta \bar{V}_n$ *for all* $n \geq N$.

Proof. Take any $(v, w) \epsilon \partial \bar{V}_n$. Let $z = (x_1, x_2, v_1, w_1, v_2, w_2, v_0, w_0)$ be the factorization of (v, w). It immediately follows that $z_\delta = \delta z$ is a weak-factorization for $(v_\delta, w_\delta) = \delta (v, w)$ and by construction this factorization uses continuation values in $\delta \bar{V}_n$. What we need to show is that there is a strong-factorization that can deliver these values also with continuations in δV. For this we will require to "create surplus" and distribute it to the players in order to fulfill the stronger incentive constraints. In the following steps, surplus is created by reestablishing the original favors (from δx to x) which gives $(1-\delta)(b-c)$. The trick is to distribute this surplus between the players in an incentive compatible way.

Without loss of generality, assume $v \geq w$, and $x_2 = 1$.

Step one. Consider the following alternative vector

$$\tilde{x} = (\delta x_1, x_2, \delta v_1, \delta w_1, \tilde{v}_2, \tilde{w}_2, \delta v_0, \delta w_0)$$

and let $(\tilde{v}, \tilde{w})$ denote the implied values. In this step we just reestablish the original favor made to player two, giving that player an extra $(1-\delta) x_2 b$ of utility.

Step two. We will now "give back" this surplus to player two through continuation values, generating slackness in its incentive constraint. First we show that there exists $g > \underline{g} > \frac{c}{b}$ such that

$$\left(\tilde{v}_2 = \delta v_2 - b(1-\delta), \tilde{w}_2 = \delta w_2 + gb(1-\delta)\right) \epsilon \delta \bar{V}_n.$$

This is done as follows: Take the original point $(v_2, w_2) \epsilon \bar{V}_n$ and without loss of generality assume it is in the boundary of $\bar{V}_n$. Consider the point $\hat{v} = v_2 - b(1-\delta)/\delta$ together with the corresponding $\hat{w}$ in the boundary of $\bar{V}_n$ and let $g = |\frac{\hat{w}-w_2}{\hat{v}-v_2}|$. Since by assumption $v > w$, and the frontier of $\bar{V}_n$ is symmetric and decreasing, $g \geq 1 \geq \underline{g} > \frac{c}{b}$. Now define

$$\tilde{w}_2 = \delta\hat{w} = \delta w_2 + g\delta \left\| \widehat{v} - v_2 \right\| = \delta w_2 + gb(1-\delta).$$

It follows by construction that $(\tilde{v}_2, \tilde{w}_2) \epsilon \delta \bar{V}_n$. In order to construct a bound independent of the point (v, w) chosen, take the values $\tilde{v}_2$ as above and $\tilde{w}_2 = \delta w_2 + \underline{g}b(1-\delta)$, lower than the above. This point is still in $\bar{V}_n$, though not in the frontier.

It is easy to verify that these continuation values and policies give values $\tilde{v} = \delta v_1$ to player one and $\tilde{w} = \delta w + \varepsilon_n$ to player two, where

$$\varepsilon_n = \frac{\alpha}{r+2\alpha}(1-\beta_n)(1-\delta)\left(\underline{g}b - c\right) \equiv (1-\beta_n)\frac{\alpha}{r+2\alpha}\varepsilon$$

where $\varepsilon = (1-\delta)\left(\underline{g}b - c\right)$..

Step 3. We will now use this *slack* and *transfer* part of it back to player one in an *incentive compatible* way. Consider first the case where $\delta x_1 \geq \frac{\varepsilon}{2b}$and set $\tilde{x}_1 = \delta x_1 - \frac{\varepsilon}{2b}$. This allocation gives values

$$\begin{pmatrix} \tilde{v} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} \delta v + (1-\beta_n)\frac{\alpha}{r+2\alpha}\frac{c\varepsilon}{2b}, \\ \delta w + (1-\beta_n)\frac{\alpha}{r+2\alpha}\frac{\varepsilon}{2} \end{pmatrix}.$$

Strong incentive compatibility requires:

$$\begin{aligned}
\tilde{w}_2 - \tilde{x}_2 c &\geq \delta w_0 \\
\tilde{w}_2 - \tilde{x}_2 c &\geq \tilde{w} \\
\tilde{v}_1 - \tilde{x}_1 c &\geq \delta v_0 \\
\tilde{v}_1 - \tilde{x}_1 c &\geq \tilde{v}.
\end{aligned}$$

As observed above, if the first and third constraint are satisfied with slack, the remaining two will hold for sufficiently high n. To check the first, note that

$$\tilde{w}_2 - \tilde{x}_2 c = \delta(w_2 - c) + \varepsilon \geq \beta_n \delta w_0 + \varepsilon$$

where the inequality follows from the weak incentive constraint $w_2 - c \geq \beta_n w_0$. Pick n sufficiently high so that $(1-\beta_n)\,\delta w_0 < \varepsilon$ and the first incentive constraint is satisfied. Since w_0 is bounded above by $\alpha b/r$, It is apparent that the two constraints for player one are satisfied for sufficiently high n regardless of the initial point (v, w).

Similar arguments can be used to verify the constraints for player one:

$$\tilde{v}_1 - \tilde{x}_1 c = \delta\,(v_1 - x_1 c) + \frac{c\varepsilon}{2b} \geq \delta\beta_n v_0 + \frac{c\varepsilon}{2b}$$

which sill exceed δv_0 for sufficiently high n.

Consider now the second case, where $\delta x_1 < \frac{\varepsilon}{2b}$. Set $x_1 = 0$ and redistribute surplus to player one by decreasing $\tilde{w}_2$ by $\Delta = \frac{\varepsilon}{2} - \frac{\delta x_1}{c} b$ and increasing $\tilde{v}_2$ by $\Delta \frac{c}{b}$. This gives another point $(\tilde{v}_2, \tilde{w}_2)$ in the set $\delta\overline{V}$. It is easy to check, following the steps used in the other case, that for large n the strong incentive constraints are satisfied. In the previous steps, we have established that $\delta\bar{V}_n \subset B_{T_n}\left(\delta\bar{V}_n\right)$ so by self-generation it follows that $\delta\bar{V}_n \subset V^*$ for sufficiently large n. $\square$

The previous lemma implies that that *int* $\bar{V} \subset V^*$ and since both $\bar{V}$ and V^* are closed, this in turn implies $\bar{V} \subset V^*$. Since we have established the reverse inclusion above, it follows that $V^* = \bar{V}$.